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Topological aspects of Abelian gauge theory in superfield formulation

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Abstract

We discuss some aspects of the topological features of a non-interacting two $(1+1)$-dimensional Abelian gauge theory in the framework of superfield formalism. This theory is described by a BRST invariant Lagrangian density in the Feynman gauge. We express the local and continuous symmetries, Lagrangian density, topological invariants and symmetric energy–momentum tensor of this theory in the language of superfields by exploiting the nilpotent (anti-)BRST and (anti-)co-BRST symmetries. In particular, the Lagrangian density and symmetric energy–momentum tensor of this topological theory turn out to be the sum of terms that geometrically correspond to the translations of some local superfields along the Grassmannian directions of the four $(2+2)$-dimensional supermanifold. In this interpretation, the (anti-)BRST and (anti-)co-BRST symmetries, that emerge after the imposition of the (dual) horizontality conditions, play a very important role.

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There are many areas of research in the modern developments of theoretical high energy physics that have brought together mathematicians as well as theoretical physicists to share their key insights into those specific fields of investigation in a constructive and illuminating manner. The subject of topological field theories (TFTs) [1–3] is one such area that has provided a meeting-ground for researchers of both varieties to enrich their understanding in a coherent and consistent fashion. Recently, the free two $(1+1)$-dimensional (2D) Abelian and self-interacting non-Abelian gauge theories (having no interaction with matter fields) have been shown [4, 5] to belong to a new class of TFTs that capture together some of the key features of Witten- and Schwarz-types of TFTs [1, 2]. Furthermore, these 2D free- as well as interacting (non-)Abelian gauge theories have been shown, in a series of papers [4–9], to represent a class of field theoretical models for the Hodge theory where symmetries of the Lagrangian density and their corresponding generators have been identified (algebraically) with the de Rham cohomology operators of differential geometry. In fact, these symmetries and corresponding generators have been exploited to establish the topological nature of the 2D free Abelian and self-interacting non-Abelian gauge theories [5]. The analogues of the above cohomological operators, in terms of the symmetries and corresponding generators, have also been found for the physical four $(3+1)$-dimensional free Abelian two-form gauge theory [10]. The geometrical interpretations for the above local and conserved generators in the context of 2D theories have also been provided [11–13] in the framework of the superfield formalism [14–18] where it has been shown that these conserved charges correspond to the translation generators along the Grassmannian (odd) as well as bosonic (even) directions of a four $(2+2)$-dimensional supermanifold. In these endeavours, a generalized version of the so-called horizontality condition [14–16] has been exploited with respect to all the three[1] super de Rham cohomology operators $(\tilde{\mathrm{d}}, \tilde{\delta}, \tilde{\Delta} = \tilde{\mathrm{d}}\tilde{\delta} + \tilde{\delta}\tilde{\mathrm{d}})$ of differential geometry defined on the $(2+2)$-dimensional supermanifold without a boundary.

In all our previous attempts [11–13] to provide a geometrical interpretation for the generators of the (anti-)BRST symmetries, (anti-)co-BRST symmetries and a bosonic symmetry in the framework of superfield formulation, we have not found a way to capture the topological features of the 2D free Abelian and self-interacting non-Abelian gauge theories (without having any interaction with matter fields). The purpose of our present paper is to show that the nilpotent $\left(s_b^2 = \bar{s}_b^2 = s_d^2 = \bar{s}_d^2 = 0\right)$ (anti-)BRST symmetries $(\bar{s}_b)s_b$ and (anti-)co-BRST symmetries $(\bar{s}_d)s_d$, Lagrangian density, topological invariants and symmetric energy–momentum tensor for the free 2D Abelian gauge theory can be expressed in terms of the superfields alone and a possible geometrical interpretation can be provided for the above physically interesting quantities in the framework of superfield formalism. We show, in particular, that the Lagrangian density and the symmetric energy-momentum tensor can be written as the sum of quantities that can be expressed in terms of the Grassmannian derivatives on the Lorentz scalar(s) and second-rank tensors, respectively. These scalar(s) and tensors are constructed from the even superfields of the theory and are found to be endowed with the proper mass dimensions. In fact, for the present TFT (i.e. 2D free Abelian gauge theory), the Lagrangian density and symmetric energy–momentum tensor turn out to have the geometrical interpretation as the sum of terms which correspond to the translations of some local (but composite) even superfields (constructed by the basic even superfields of the theory) along the Grassmannian directions of the supermanifold. In a similar fashion, the zero-forms of the topological invariants of this theory turn out to be translations of the local (but composite) even superfields (constructed by the basic odd superfields of the theory) along the Grassmannian directions of the $(2+2)$-dimensional supermanifold. These translations are generated by the conserved and nilpotent (anti-)BRST and (anti-)co-BRST charges. One of the key features of this TFT is the fact that the Lagrangian density and energy–momentum tensor can be expressed in terms of the even superfields alone and the (anti-)BRST and (anti-)co-BRST transformations act on the $\theta\bar{\theta}$-components of the one and the same combinations of the even superfields. The symmetric nature of the energy–momentum tensor comes out very naturally in the framework of superfield formulation. In the above derivations, the (dual) horizontality conditions w.r.t. super cohomological operators $\tilde{\mathrm{d}}$ and $\tilde{\delta}$ play a very significant role. These conditions are, of course, required for the derivations of the (anti-)BRST- and (anti-)co-BRST symmetries which, in turn, provide the geometrical interpretation for their generators as the 'translation generators' along the Grassmannian (θ and $\bar{\theta}$) directions of the supermanifold.

[1] On an ordinary flat manifold without a boundary, a set $(\mathrm{d}, \delta, \Delta)$ of three cohomological operators can be defined which obey the algebra: $\mathrm{d}^2 = \delta^2 = 0$, $\Delta = (\mathrm{d}+\delta)^2 = \mathrm{d}\delta + \delta\mathrm{d} \equiv \{\mathrm{d}, \delta\}$, $[\Delta, \mathrm{d}] = [\Delta, \delta] = 0$ where $\mathrm{d} = \mathrm{d}x^\mu \partial_\mu$ and $\delta = \pm * \mathrm{d} *$ (with $*$ as the Hodge duality operation) are the nilpotent (of order 2) exterior- and co-exterior derivatives and Δ is the Laplacian operator [19–22].

The superfield formulation of the above theory also sheds light on some new symmetries of the Lagrangian density and symmetric energy–momentum tensor (see, e.g., equations (19*b*) and (38)) which were not known hitherto in our previous studies in the framework of Lagrangian formalism [4–10].

Let us begin with the BRST invariant Lagrangian density $\mathcal{L}_b$ for the free two (1 + 1)-dimensional[2] Abelian gauge theory in the Feynman gauge [23–26]

$$\mathcal{L}_b = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial\cdot A)^2 - \mathrm{i}\partial_\mu\bar{C}\partial^\mu C \equiv \frac{1}{2}E^2 - \frac{1}{2}(\partial\cdot A)^2 - \mathrm{i}\partial_\mu\bar{C}\partial^\mu C \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor derived from the connection one-form $A = \mathrm{d}x^\mu A_\mu$ (with A_μ as the vector potential) by application of the exterior derivative $\mathrm{d} = \mathrm{d}x^\mu\partial_\mu$(with $\mathrm{d}^2 = 0$) as $F = \mathrm{d}A = \frac{1}{2}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)F_{\mu\nu}$. The gauge-fixing term is derived as $\delta A = (\partial\cdot A)$ where $\delta = -*\,\mathrm{d}*$ (with $\delta^2 = 0$) is the co-exterior derivative and $*$ is the Hodge duality operation. The (anti-)commuting ($C\bar{C} + \bar{C}C = 0$, $C^2 = \bar{C}^2 = 0$) (anti-)ghost fields $(\bar{C})C$ are required in the theory to maintain unitarity and gauge invariance together. The above Lagrangian density (1) respects the following on-shell ($\Box C = \Box\bar{C} = 0$) nilpotent $\left(s_b^2 = 0, s_d^2 = 0\right)$ BRST (s_b)[3] and dual(co)-BRST (s_d) symmetry transformations [4–9]

$$\begin{aligned} &s_b A_\mu = \partial_\mu C \qquad && s_b C = 0 \qquad && s_b\bar{C} = -\mathrm{i}(\partial\cdot A)\\ &s_d A_\mu = -\varepsilon_{\mu\nu}\partial^\nu\bar{C} \qquad && s_d\bar{C} = 0 \qquad && s_d C = -\mathrm{i}E. \end{aligned} \tag{2}$$

The Lagrangian density (1) is also invariant under the on-shell anti-BRST $(\bar{s}_b)$ (with $s_b\bar{s}_b + \bar{s}_b s_b = 0$) and anti-co-BRST $(\bar{s}_d)$ (with $s_d\bar{s}_d + \bar{s}_d s_d = 0$) symmetries

$$\begin{aligned} &\bar{s}_b A_\mu = \partial_\mu\bar{C} \qquad && \bar{s}_b\bar{C} = 0 \qquad && \bar{s}_b C = +\mathrm{i}(\partial\cdot A)\\ &\bar{s}_d A_\mu = -\varepsilon_{\mu\nu}\partial^\nu C \qquad && \bar{s}_d C = 0 \qquad && \bar{s}_d\bar{C} = +\mathrm{i}E. \end{aligned} \tag{3}$$

The anti-commutator of these nilpotent, local, continuous and covariant symmetries (i.e. $s_w = \{s_b, s_d\} = \{\bar{s}_b, \bar{s}_d\}$) leads to a bosonic symmetry[4] s_w $\left(s_w^2 \neq 0\right)$ transformations [4–9]

$$s_w A_\mu = \partial_\mu E - \varepsilon_{\mu\nu}\partial^\nu(\partial\cdot A) \qquad s_w C = 0 \qquad s_w\bar{C} = 0 \tag{4}$$

under which the Lagrangian density (1) transforms to a total derivative. All the above continuous symmetry transformations can be concisely (and succinctly) expressed, in terms of the Noether conserved charges Q_r and $\bar{Q}_r$ [4–9], as

$$\begin{aligned} &s_r\Psi = -\mathrm{i}[\Psi, Q_r]_\pm \qquad && Q_r = Q_b, Q_d, Q_w, Q_g\\ &\bar{s}_r\Psi = -\mathrm{i}[\Psi, \bar{Q}_r]_\pm \qquad && \bar{Q}_r = \bar{Q}_b, \bar{Q}_d \end{aligned} \tag{5}$$

where brackets $[\ ,\ ]_\pm$ stand for the (anti-)commutators for any arbitrary generic field Ψ being (fermionic)bosonic in nature. Here, the conserved ghost charge Q_g generates the continuous scale transformations: $C \to \mathrm{e}^{-\Sigma}C$, $\bar{C} \to \mathrm{e}^{\Sigma}\bar{C}$, $A_\mu \to A_\mu$ (where Σ is a global parameter). The local field theoretical expressions for Q_r and $\bar{Q}_r$ (which are not required for our present discussion) are given in [4–9].

The geometrical interpretation for the local and conserved (anti-)BRST $(\bar{Q}_b)Q_b$ and (anti-)co-BRST $(\bar{Q}_d)Q_d$ charges as the translation generators along the Grassmannian

[2] We follow here the conventions and notations such that the 2D flat Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}\,(+1, -1)$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = (\partial_0)^2 - (\partial_1)^2$, $F_{01} = -\varepsilon^{\mu\nu}\partial_\mu A_\nu = E = \partial_0 A_1 - \partial_1 A_0$, $\varepsilon_{01} = \varepsilon^{10} = +1$, $\varepsilon^{\mu\rho}\varepsilon_{\rho\nu} = \delta^\mu_\nu$. Here Greek indices $\mu, \nu, \ldots = 0, 1$ correspond to the spacetime directions on the 2D manifold.

[3] We adopt here the notations and conventions of [26]. In fact, in its full glory, a nilpotent $(\delta^2_{(D)B} = 0)$ (co-)BRST transformation $(\delta_{(D)B})$ is equivalent to the product of an anti-commuting $(\eta C = -C\eta, \eta\bar{C} = -\bar{C}\eta)$ spacetime-independent parameter η and $(s_d)s_b$ (i.e. $\delta_{(D)B} = \eta s_{(d)b}$) where $s^2_{(d)b} = 0$.

[4] This symmetry has not been discussed in [27] where the nilpotent transformations (2) and (3) have been discussed on a closed 2D Riemann surface. We thank Professor N Nakanishi for some critical and constructive comments on our earlier works and for bringing to our notice [27].

directions of the $(2+2)$-dimensional supermanifold has been shown [11–13] in the framework of superfield formulation [14–18] where the even (bosonic) superfield $B_\mu(x,\theta,\bar\theta)$, and odd (fermionic) fields $\Phi(x,\theta,\bar\theta)$ and $\bar\Phi(x,\theta,\bar\theta)$ have been expanded in terms of the super coordinates $(x,\theta,\bar\theta)$, the dynamical fields of the Lagrangian density (1) and some extra (secondary) fields (e.g., $R_\mu(x)$, $\bar R_\mu(x)$, $S_\mu(x)$, $s(x)$, $\bar s(x)$) as given below [11]:

$$\begin{aligned} B_\mu(x,\theta,\bar\theta) &= A_\mu(x) + \theta\bar R_\mu(x) + \bar\theta R_\mu(x) + \mathrm{i}\theta\bar\theta S_\mu(x) \\ \Phi(x,\theta,\bar\theta) &= C(x) + \mathrm{i}\theta(\partial\cdot A)(x) - \mathrm{i}\bar\theta E(x) + \mathrm{i}\theta\bar\theta s(x) \\ \bar\Phi(x,\theta,\bar\theta) &= \bar C(x) + \mathrm{i}\theta E(x) - \mathrm{i}\bar\theta(\partial\cdot A)(x) + \mathrm{i}\theta\bar\theta \bar s(x). \end{aligned} \tag{6}$$

Here, some of the noteworthy points are: (i) the $(2+2)$-dimensional supermanifold is parametrized by the superspace coordinates $Z^M = (x^\mu,\theta,\bar\theta)$ where $x^\mu(\mu = 0,1)$ are the two even (bosonic) spacetime coordinates and, θ and $\bar\theta$ are the two odd (Grassmannian) coordinates (with $\theta^2 = \bar\theta^2 = 0, \theta\bar\theta + \bar\theta\theta = 0$). (ii) The expansions are along the odd (fermionic) superspace coordinates θ and $\bar\theta$ and even (bosonic) $(\theta\bar\theta)$ directions of the supermanifold. (iii) All the fields are local functions of the spacetime coordinates x^μ alone (i.e. $A_\mu(x,0,0) = A_\mu(x), C(x,0,0) = C(x)$ etc). Now the horizontality[5] condition [14–16] on the super curvature (two-form) tensor $\tilde F = \tilde{\mathrm{d}}\tilde A$ for the Abelian gauge theory

$$\tilde F = \tfrac{1}{2}(\mathrm{d}Z^M\wedge \mathrm{d}Z^N)\,\tilde F_{MN} = \tilde{\mathrm{d}}\tilde A \equiv \mathrm{d}A = \tfrac{1}{2}(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu)\,F_{\mu\nu} = F \tag{7}$$

leads to the following expressions for the extra (secondary) fields [11]:

$$\begin{aligned} &R_\mu(x) = \partial_\mu C(x) && \bar R_\mu(x) = \partial_\mu \bar C(x) && s(x) = 0 \\ &S_\mu(x) = -\partial_\mu[(\partial\cdot A)](x) && E(x) = 0 && \bar s(x) = 0 \end{aligned} \tag{8}$$

in terms of the basic fields (cf equation (1)) of the theory. The super curvature tensor $\tilde F$ is constructed by the super exterior derivative $\tilde{\mathrm{d}}$ and super connection one-form $\tilde A$, defined on the $(2+2)$-dimensional supermanifold, as

$$\begin{aligned} \tilde{\mathrm{d}} &= \mathrm{d}Z^M\partial_M = \mathrm{d}x^\mu\partial_\mu + \mathrm{d}\theta\partial_\theta + \mathrm{d}\bar\theta\partial_{\bar\theta} \\ \tilde A &= \mathrm{d}Z^M\tilde A_M = \mathrm{d}x^\mu B_\mu(x,\theta,\bar\theta) + \mathrm{d}\theta\bar\Phi(x,\theta,\bar\theta) + \mathrm{d}\bar\theta\Phi(x,\theta,\bar\theta). \end{aligned} \tag{9}$$

The substitution of (8) into expansion (6) leads to the following:

$$\begin{aligned} B_\mu(x,\theta,\bar\theta) &= A_\mu(x) + \theta\partial_\mu\bar C(x) + \bar\theta\partial_\mu C(x) - \mathrm{i}\theta\bar\theta\partial_\mu(\partial\cdot A)(x) \\ &\equiv A_\mu(x) + \theta(\bar s_b A_\mu(x)) + \bar\theta(s_b A_\mu(x)) + \theta\bar\theta(s_b\bar s_b A_\mu(x)) \\ \Phi(x,\theta,\bar\theta) &= C(x) + \mathrm{i}\theta(\partial\cdot A)(x) \equiv C(x) + \theta(\bar s_b C(x)) \\ \bar\Phi(x,\theta,\bar\theta) &= \bar C(x) - \mathrm{i}\bar\theta(\partial\cdot A)(x) \equiv \bar C(x) + \bar\theta(s_b\bar C(x)). \end{aligned} \tag{10a}$$

Thus, we note that the horizontality condition in (7) leads to (i) the derivation of secondary fields in terms of the basic fields of the Lagrangian density. (ii) The (anti-)BRST symmetry transformations for the Lagrangian density listed in (2) and (3). (iii) Geometrical interpretation for the (anti-)BRST charges $(\bar Q_b)Q_b$ as the translation generators along the Grassmannian directions of the $(2+2)$-dimensional supermanifold, i.e.,

$$\begin{aligned} &\lim_{\theta,\bar\theta\to 0}\frac{\partial}{\partial\bar\theta}B_\mu = \mathrm{i}[Q_b, A_\mu] \equiv s_b A_\mu && \lim_{\theta,\bar\theta\to 0}\frac{\partial}{\partial\theta}B_\mu = \mathrm{i}[\bar Q_b, A_\mu] \equiv \bar s_b A_\mu \\ &\lim_{\theta,\bar\theta\to 0}\frac{\partial}{\partial\bar\theta}\Phi = -\mathrm{i}\{Q_b, C\} \equiv s_b C && \lim_{\theta,\bar\theta\to 0}\frac{\partial}{\partial\theta}\Phi = -\mathrm{i}\{\bar Q_b, C\} \equiv \bar s_b C \\ &\lim_{\theta,\bar\theta\to 0}\frac{\partial}{\partial\bar\theta}\bar\Phi = -\mathrm{i}\{Q_b, \bar C\} \equiv s_b \bar C && \lim_{\theta,\bar\theta\to 0}\frac{\partial}{\partial\theta}\bar\Phi = -\mathrm{i}\{\bar Q_b, \bar C\} \equiv \bar s_b \bar C \end{aligned} \tag{10b}$$

[5] This condition is referred to as the 'soul-flatness' condition by Nakanishi and Ojima in [23].

as is evident from equations (5) and (10a). It will be noted here that we have taken the translation generators along the θ and $\bar\theta$ directions of the supermanifold as $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\bar\theta}$, respectively. (iv) The nilpotent (anti-)BRST transformations $(\bar s_b)s_b$ are along the Grassmannian directions $(\theta)\bar\theta$. (v) There is a mapping between super exterior derivative $\tilde{\mathrm{d}}$ and the (anti-)BRST charges as $\tilde{\mathrm{d}} \Leftrightarrow (Q_b, \bar Q_b)$. (vi) It is useful and interesting (for later convenience) to note that now the nilpotent (anti-)BRST symmetries of equations (2) and (3) can be rewritten in terms of the superfields as

$$
\begin{aligned}
&s_b B_\mu(x,\theta,\bar\theta) = \partial_\mu \Phi(x,\theta,\bar\theta) \qquad s_b \Phi(x,\theta,\bar\theta) = 0 \qquad s_b \bar\Phi(x,\theta,\bar\theta) = -\mathrm{i}(\partial\cdot B)(x,\theta,\bar\theta)\\
&\bar s_b B_\mu(x,\theta,\bar\theta) = \partial_\mu \bar\Phi(x,\theta,\bar\theta) \qquad \bar s_b \bar\Phi(x,\theta,\bar\theta) = 0 \qquad \bar s_b \Phi(x,\theta,\bar\theta) = +\mathrm{i}(\partial\cdot B)(x,\theta,\bar\theta)
\end{aligned}
\tag{11}
$$

where the expansions (10a) which emerge after the application of the horizontality condition w.r.t. the super exterior derivative $\tilde{\mathrm{d}}$ are taken into account. The sanctity and correctness of the above equation can be checked easily by first applying the transformations w.r.t. $\delta_B = \eta\, s_{\mathrm{b}}$, and then, rederiving transformations s_{b} from it.

The analogue[6] of the horizontality condition (7) w.r.t. the super co-exterior derivative $\tilde\delta = -\star\,\tilde{\mathrm{d}}\star$ and its operation on the super one-form connection $\tilde A$, namely

$$
\begin{aligned}
&\tilde\delta\tilde A = \delta A \qquad \delta = -*\,\mathrm{d}\,* \qquad A = \mathrm{d}x^\mu A_\mu \qquad \delta A = (\partial\cdot A)\\
&\tilde\delta\tilde A = (\partial_\mu B^\mu) + s^{\theta\theta}(\partial_\theta\Phi) + s^{\bar\theta\bar\theta}(\partial_{\bar\theta}\bar\Phi) + s^{\theta\bar\theta}(\partial_\theta\bar\Phi + \partial_{\bar\theta}\Phi) - \varepsilon^{\mu\theta}(\partial_\mu\Phi + \varepsilon_{\mu\nu}\partial_\theta B^\nu)\\
&\qquad\quad - \varepsilon^{\mu\bar\theta}(\partial_\mu\bar\Phi + \varepsilon_{\mu\nu}\partial_{\bar\theta}B^\nu)
\end{aligned}
\tag{12}
$$

leads to the following expression for the secondary (extra) fields in terms of the basic fields of the Lagrangian density (1) for the theory [11, 12]:

$$
\begin{aligned}
&R_\mu(x) = -\varepsilon_{\mu\nu}\partial^\nu\bar C(x) \qquad \bar R_\mu(x) = -\varepsilon_{\mu\nu}\partial^\nu C(x) \qquad s(x) = 0\\
&S_\mu(x) = +\varepsilon_{\mu\nu}\partial^\nu E(x) \qquad \bar s(x) = 0 \qquad (\partial\cdot A)(x) = 0.
\end{aligned}
\tag{13}
$$

In the above computations, the Hodge duality $\star$ operation on the super differentials $(\mathrm{d}Z^M)$ and their wedge products $(\mathrm{d}Z^M\wedge\mathrm{d}Z^N)$, defined on the (2 + 2)-dimensional supermanifold, is

$$
\begin{aligned}
&\star(\mathrm{d}x^\mu) = \varepsilon^{\mu\nu}(\mathrm{d}x_\nu) \qquad \star(\mathrm{d}\theta) = (\mathrm{d}\bar\theta) \qquad \star(\mathrm{d}\bar\theta) = (\mathrm{d}\theta)\\
&\star(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu) = \varepsilon^{\mu\nu} \qquad \star(\mathrm{d}x^\mu\wedge\mathrm{d}\theta) = \varepsilon^{\mu\theta} \qquad \star(\mathrm{d}x^\mu\wedge\mathrm{d}\bar\theta) = \varepsilon^{\mu\bar\theta}\\
&\star(\mathrm{d}\theta\wedge\mathrm{d}\theta) = s^{\theta\theta} \qquad \star(\mathrm{d}\theta\wedge\mathrm{d}\bar\theta) = s^{\theta\bar\theta} \qquad \star(\mathrm{d}\bar\theta\wedge\mathrm{d}\bar\theta) = s^{\bar\theta\bar\theta}
\end{aligned}
\tag{14}
$$

where $\varepsilon^{\mu\theta} = -\varepsilon^{\theta\mu}$, $\varepsilon^{\mu\bar\theta} = -\varepsilon^{\bar\theta\mu}$, $s^{\theta\bar\theta} = s^{\bar\theta\theta}$ etc. In terms of the expressions (13), the super expansion (6) can be re-expressed as

$$
\begin{aligned}
B_\mu(x,\theta,\bar\theta) &= A_\mu(x) - \theta\varepsilon_{\mu\nu}\partial^\nu C(x) - \bar\theta\varepsilon_{\mu\nu}\partial^\nu\bar C(x) + \mathrm{i}\theta\bar\theta\varepsilon_{\mu\nu}\partial^\nu E(x)\\
&\equiv A_\mu(x) + \theta(\bar s_d A_\mu(x)) + \bar\theta(s_d A_\mu(x)) + \theta\bar\theta(s_d\bar s_d A_\mu(x))\\
\Phi(x,\theta,\bar\theta) &= C(x) - \mathrm{i}\bar\theta E(x) \equiv C(x) + \bar\theta(s_d C(x))\\
\bar\Phi(x,\theta,\bar\theta) &= \bar C(x) + \mathrm{i}\theta E(x) \equiv \bar C(x) + \theta(\bar s_d\bar C(x)).
\end{aligned}
\tag{15}
$$

We pin-point some of the salient features of the nilpotent (anti-)co-BRST symmetry transformations *vis-à-vis* (anti-)BRST symmetry transformations (and their generators). The common features are: (i) the (anti-)BRST and (anti-)co-BRST symmetry transformations are generated along the $\theta(\bar\theta)$ directions of the supermanifold. (ii) geometrically, the translation

[6] Henceforth, this condition w.r.t. (super) co-exterior derivatives will be called the dual horizontality condition because $(\tilde\delta)\delta$ and $(\tilde{\mathrm{d}})\mathrm{d}$ are Hodge dual to each other on the (super) manifold.

generators along the Grassmannian directions of the supermanifold are the conserved and nilpotent (anti-)BRST and (anti-)co-BRST charges (cf equation (5)). (iii) for the odd (fermionic) superfields, the translations are either along θ or $\bar{\theta}$ directions for in the case of (anti-)BRST and (anti-)co-BRST symmetries, respectively. (iv) for the bosonic superfield, the translations are along both θ as well as $\bar{\theta}$ directions when we consider (anti-)BRST and/or (anti-)co-BRST symmetries. The key differences are: (i) comparison between (10*a*) and (15) shows that the (anti-)BRST transformations generate translations along $(\theta)\bar{\theta}$ directions for the odd fields $(C)\bar{C}$. In contrast, for the same fields, the (anti-)co-BRST transformations generate translations along $(\bar{\theta})\theta$ directions of the supermanifold. (ii) the restrictions $\tilde{\delta}\tilde{A} = \delta A$ and $\tilde{\mathrm{d}}\tilde{A} = \mathrm{d}A$ (w.r.t. different cohomological operators) produce (anti-)co-BRST and (anti-) BRST symmetry transformations. (iii) the expressions for R_μ and $\bar{R}_\mu$ in (8) and (13) are such that the kinetic energy and gauge-fixing terms of (1) remain invariant under (anti-)BRST and (anti-)co-BRST symmetries, respectively. (iv) it is very interesting to note that the nilpotent (anti-)co-BRST transformations in (2) and (3) can now be re-expressed in terms of the superfields (analogous to equation (11)) as

$$\begin{array}{ll} s_d B_\mu(x,\theta,\bar{\theta}) = -\varepsilon_{\mu\nu}\partial^\nu\bar{\Phi}(x,\theta,\bar{\theta}) & s_d\bar{\Phi}(x,\theta,\bar{\theta}) = 0 \\ s_d\Phi(x,\theta,\bar{\theta}) = +\mathrm{i}\varepsilon^{\mu\nu}\partial_\mu B_\nu(x,\theta,\bar{\theta}) & \bar{s}_d\Phi(x,\theta,\bar{\theta}) = 0 \\ \bar{s}_d B_\mu(x,\theta,\bar{\theta}) = -\varepsilon_{\mu\nu}\partial^\nu\Phi(x,\theta,\bar{\theta}) & \bar{s}_d\bar{\Phi}(x,\theta,\bar{\theta}) = -\mathrm{i}\varepsilon^{\mu\nu}\partial_\mu B_\nu(x,\theta,\bar{\theta}) \end{array} \tag{16}$$

where expansions (15) have been taken into account, which are obtained after the imposition of the dual horizontality condition with respect to the super co-exterior derivative $\tilde{\delta}$. (v) for the (anti-)BRST and (anti-)co-BRST symmetries, the mapping are: $\tilde{\mathrm{d}} \Leftrightarrow (Q_b, \bar{Q}_b), \tilde{\delta} \Leftrightarrow (Q_d, \bar{Q}_d)$, but the ordinary exterior and co-exterior derivatives d and δ are identified with $(Q_b, \bar{Q}_d)$ and $(Q_d, \bar{Q}_b)$ because of the ghost number considerations of a typical state in the quantum Hilbert space [4–7].

Exploiting equations (2), (3) and (5), it can be checked that the Lagrangian density (1) can be expressed, modulo some total derivatives, as

$$\begin{aligned} \mathcal{L}_b &= \{Q_d, T_1\} + \{Q_b, T_2\} \equiv \{\bar{Q}_d, P_1\} + \{\bar{Q}_b, P_2\} \\ \mathcal{L}_b &= s_d(\mathrm{i}T_1) + s_b(\mathrm{i}T_2) + \partial_\mu Y^\mu \equiv \bar{s}_d(\mathrm{i}P_1) + \bar{s}_b(\mathrm{i}P_2) + \partial_\mu Y^\mu \end{aligned} \tag{17}$$

where $T_1 = \frac{1}{2}(EC), T_2 = -\frac{1}{2}((\partial\cdot A)\bar{C}), P_1 = -\frac{1}{2}(E\bar{C}), P_2 = \frac{1}{2}((\partial\cdot A)C)$ and $Y^\mu = \frac{\mathrm{i}}{2}(\partial^\mu\bar{C}C - \partial^\mu C\bar{C})$. *The above Lagrangian density can also be understood as translations, generated by the (anti-)BRST and (anti-)co-BRST charges, along the Grassmannian (θ and $\bar{\theta}$) directions* of the supermanifold as given below:

$$\mathcal{L}_b = \frac{\mathrm{i}}{2}\frac{\partial}{\partial\theta}[\{(\varepsilon^{\mu\nu}\partial_\mu B_\nu)\bar{\Phi}\}|_{\text{(anti-)co-BRST}} + \{(\partial\cdot B)\Phi\}|_{\text{(anti-)BRST}}] \tag{18a}$$

$$\mathcal{L}_b = -\frac{\mathrm{i}}{2}\frac{\partial}{\partial\bar{\theta}}[\{(\varepsilon^{\mu\nu}\partial_\mu B_\nu)\Phi\}|_{\text{(anti-)co-BRST}} + \{(\partial\cdot B)\bar{\Phi}\}|_{\text{(anti-)BRST}}] \tag{18b}$$

where the subscripts (anti-)BRST and (anti-)co-BRST stand for the insertion of the expansions given in equations (10*a*) and (15), respectively. It is obvious that the expression for the Lagrangian density $\mathcal{L}_b = \{\bar{Q}_d, P_1\} + \{\bar{Q}_b, P_2\}$ of equation (17) is captured by (18*a*) and $\mathcal{L}_b = \{Q_d, T_1\} + \{Q_b, T_2\}$ is captured by (18*b*) in the language of the derivatives on the composite superfields defined on the supermanifold. Geometrically, (18*a*) implies the translation (by the translation generator $\frac{\partial}{\partial\theta}$) of the composite superfields $(\varepsilon^{\mu\nu}\partial_\mu B_\nu)\bar{\Phi}$ and $(\partial\cdot B)\Phi$ along the θ-direction of the supermanifold. For this interpretation, the nilpotent (anti-)BRST and (anti-)co-BRST symmetries, which emerge after the imposition of the (dual) horizontality conditions with respect to the super cohomological operator(s) $\tilde{\mathrm{d}}$ (and $\tilde{\delta}$), play an important role. Similar interpretation can be associated with (18*b*) as well. In terms of the superfield expansion in (6), we can re-express the Lagrangian density (1) (or (17)) as

$$\begin{aligned}\mathcal{L}_b = \frac{\mathrm{i}}{4}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}[B_\mu(x,\theta,\bar{\theta})B^\mu(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} \\ + \frac{\mathrm{i}}{4}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}[B_\mu(x,\theta,\bar{\theta})B^\mu(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}} \\ \equiv -\frac{1}{2}[\mathrm{i}\bar{R}_\mu R^\mu + A^\mu S_\mu]|_{\text{(anti-)BRST}} - \frac{1}{2}[\mathrm{i}\bar{R}_\mu R^\mu + A^\mu S_\mu]|_{\text{(anti-)co-BRST}}\end{aligned} \tag{19a}$$

which turns out, in the language of symmetry transformations, to be equivalent to

$$\mathcal{L}_b = \frac{\mathrm{i}}{4}s_b\bar{s}_b(A_\mu(x)A^\mu(x)) + \frac{\mathrm{i}}{4}s_d\bar{s}_d(A_\mu(x)A^\mu(x)). \tag{19b}$$

The subscripts (anti-)BRST and (anti-)co-BRST in (19*a*) stand for the insertion of the results from equations (8) and (13), respectively. In fact, the Lagrangian densities in (19*a*) and (19*b*) differ from the Lagrangian density (1) by a total derivative: $\frac{1}{2}\partial^\mu[A_\mu(\partial\cdot A) + \varepsilon_{\mu\nu}A^\nu E]$. A few comments are in order. First, it is evident that the $(\theta\bar{\theta})$-component in the expansion of the product $B_\mu(x,\theta,\bar{\theta})B^\mu(x,\theta,\bar{\theta})$ leads to the derivation of the Lagrangian density (1) as the sum of terms on which the Grassmannian derivatives operate. Over and above, one has to exploit the (anti-)BRST and (anti-)co-BRST symmetries to obtain the exact expression for the Lagrangian density (modulo some total derivatives). Second, the horizontality condition (7) and its analogue in (12) play a very important role in the above derivation. Third, the geometrical interpretation for the Lagrangian density (19*a*) can be thought of as being equivalent to a couple of successive translations for the Lorentz super-scalar $B_\mu(x,\theta,\bar{\theta})B^\mu(x,\theta,\bar{\theta})$ along the θ and $\bar{\theta}$ directions of the supermanifold. Finally, *it appears to be an essential feature of a TFT that the Lagrangian density can be expressed as the $\theta\bar{\theta}$-component of a Lorentz super-scalar that can be constructed by the even superfields of the theory. On this scalar, one has to apply (anti-)BRST and (anti-)co-BRST symmetries that emerge after the imposition of the (dual) horizontality conditions*.

Now let us concentrate on the topological invariants of the theory. For the ordinary 2D manifold [7], there are four sets of such invariants w.r.t. conserved $(\dot{Q}_b = \dot{\bar{Q}}_b = \dot{Q}_d = \dot{\bar{Q}}_d = 0)$ and on-shell $(\Box C = \Box\bar{C} = 0)$ nilpotent $\left(Q_b^2 = \bar{Q}_b^2 = Q_d^2 = \bar{Q}_d^2 = 0\right)$ (anti-)BRST and (anti-) co-BRST charges. These are (for $k = 0, 1, 2$)

$$I_k = \oint_{C_k} V_k \qquad \bar{I}_k = \oint_{C_k} \bar{V}_k \qquad J_k = \oint_{C_k} W_k \qquad \bar{J}_k = \oint_{C_k} \bar{W}_k \tag{20}$$

where C_k are the k-dimensional homology cycles in the 2D manifold and $(\bar{V}_k)V_k$ and $(\bar{W}_k)W_k$ are the k (=0, 1, 2)-forms which are constructed w.r.t. (anti-)BRST charges $(\bar{Q}_b)Q_b$ and (anti-)co-BRST charges $(\bar{Q}_d)Q_d$, respectively. The forms V_k w.r.t. the nilpotent $\left(Q_b^2 = 0\right)$ and conserved $(\dot{Q}_b = 0)$ BRST charge Q_b are [5–7]

$$\begin{aligned}&V_0 = -(\partial\cdot A)C \qquad V_1 = [\mathrm{i}C\partial_\mu\bar{C} - (\partial\cdot A)A_\mu]\,\mathrm{d}x^\mu \\ &V_2 = \mathrm{i}[A_\mu\partial_\nu\bar{C} - \bar{C}\partial_\mu A_\nu]\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu.\end{aligned} \tag{21}$$

[7] The 2D Minkowskian manifold is actually a non-compact manifold. Thus, for the precise and accurate meaning of the topological invariants, homology cycles, etc, one has to consider the Euclidean version of the 2D Minkowskian manifold which turns out to be a closed 2D Riemann surface. Now the Greek indices $\mu, \nu, \rho, \ldots = 1, 2$ will imply the Euclidean directions and the flat metric on this manifold will carry the same signs (unlike the opposite signs for the Minkowskian manifold). Such kinds of analyses have been performed in [27] for the 2D (non-)Abelian gauge theories. For the sake of brevity, however, we shall continue with the Minkowskian notations but shall keep in mind this important fact and crucial point.

It is straightforward to check that forms $\bar{V}_k$ w.r.t. anti-BRST charge $\bar{Q}_b$ can be obtained from the above by exploiting the discrete symmetry transformations $C \leftrightarrow \bar{C}$, $(\partial \cdot A) \leftrightarrow -(\partial \cdot A)$ that connect BRST and anti-BRST transformations in (2) and (3). The forms W_k w.r.t. the co-BRST charge Q_d are [5–7]

$$\begin{aligned} W_0 &= E\bar{C} \qquad W_1 = [\bar{C}\varepsilon_{\mu\rho}\partial^\rho C - \mathrm{i}EA_\mu]\,\mathrm{d}x^\mu \\ W_2 &= \mathrm{i}\left[\varepsilon_{\mu\rho}\partial^\rho C A_\nu + \frac{C}{2}\varepsilon_{\mu\nu}(\partial \cdot A)\right]\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \end{aligned} \tag{22}$$

and $\bar{W}_k$ can be obtained from the above by the discrete symmetry transformations: $C \leftrightarrow \bar{C}$, $E \leftrightarrow -E$ under which (anti-)co-BRST transformations in (2) and (3) are connected with each other. In the language of the superfields $B_\mu(x, \theta, \bar{\theta})$, $\Phi(x, \theta, \bar{\theta})$, $\bar{\Phi}(x, \theta, \bar{\theta})$, the topological invariants in (21) can be recast as the θ and $\bar{\theta}$ independent components in

$$\begin{aligned} V_0 &= -(\partial \cdot B)\Phi \qquad V_1 = [\mathrm{i}\Phi\partial_\mu\bar{\Phi} - (\partial \cdot B)B_\mu]\,\mathrm{d}x^\mu \\ V_2 &= \mathrm{i}[B_\mu\partial_\nu\bar{\Phi} - \bar{\Phi}\partial_\mu B_\nu]\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \end{aligned} \tag{23}$$

where we have to use the on-shell conditions $\Box\,\Phi = \Box\,\bar{\Phi} = 0, \Box\,B_\mu = 0$ (which imply the validity of all the equations of motion $\Box\,C = \Box\,\bar{C} = \Box\,A_\mu = \Box(\partial \cdot A) = \Box\,E = 0$ for the Lagrangian density (1)). Furthermore, we have to use expansions (10*a*) which are obtained after the imposition of the horizontality condition (7). In fact, we notice here that, to obtain the expressions for the topological invariants of the theory w.r.t. (anti-)BRST charges $(\bar{Q}_b)Q_b$ and (anti-)co-BRST charges $(\bar{Q}_d)Q_d$ in terms of superfields, all one has to do is to replace:

$$\begin{aligned} &C \to \Phi \qquad \bar{C} \to \bar{\Phi} \qquad A_\mu \to B_\mu \qquad (\partial \cdot A) \to (\partial \cdot B) \\ &E = -\varepsilon^{\mu\nu}\partial_\mu A_\nu \to -\varepsilon^{\mu\nu}\partial_\mu B_\nu. \end{aligned} \tag{24}$$

This straightforward substitution yields the desired results here because the expansions in (10*a*) and (15) (after the imposition of constraints $\tilde{\mathrm{d}}\tilde{A} = \mathrm{d}A$ and $\tilde{\delta}\tilde{A} = \delta A$) are such that the analogue of transformations (2) and (3) are exactly imitated in terms of superfields in equations (11) and (16), respectively. Even the on-shell ($\Box\,\Phi = \Box\,\bar{\Phi} = 0$) nilpotent properties of the (anti-)co-BRST and (anti-)BRST transformations in (16) and (11) are the same as that of the ordinary ghost fields (i.e. $\Box\,C = \Box\,\bar{C} = 0$). It is illuminating, however, to check that the zero-forms $(\bar{V}_0)V_0$ and $(\bar{W}_0)W_0$ w.r.t. (anti-)BRST and (anti-)co-BRST charges can be computed directly from the expansion of the product of the superfields $\Phi(x, \theta, \bar{\theta})\bar{\Phi}(x, \theta, \bar{\theta})$ along the $\theta, \bar{\theta}$ and $\theta\bar{\theta}$ directions, namely,

$$\begin{aligned} (\Phi\bar{\Phi})|_{\text{(anti-)BRST}} &= C\bar{C} + \mathrm{i}\theta\bar{C}(\partial \cdot A) + \mathrm{i}\bar{\theta}C(\partial \cdot A) + \theta\bar{\theta}(\partial \cdot A)^2 \\ (\Phi\bar{\Phi})|_{\text{(anti-)co-BRST}} &= C\bar{C} - \mathrm{i}\theta C E - \mathrm{i}\bar{\theta}\bar{C}E - \theta\bar{\theta}E^2 \end{aligned} \tag{25}$$

where the subscripts stand for the expansions in (10*a*) and (15) that are obtained after the imposition of the horizontality and the analogue of horizontality conditions in (7) and (12), respectively. Now, it is straightforward to check that

$$\begin{aligned} &\mathrm{i}\frac{\partial(\Phi\bar{\Phi})|_{\text{(anti-)BRST}}}{\partial\theta} = \bar{V}_0 \qquad \mathrm{i}\frac{\partial(\Phi\bar{\Phi})|_{\text{(anti-)BRST}}}{\partial\bar{\theta}} = V_0 \\ &\mathrm{i}\frac{\partial(\Phi\bar{\Phi})|_{\text{(anti-)co-BRST}}}{\partial\theta} = \bar{W}_0 \qquad \mathrm{i}\frac{\partial(\Phi\bar{\Phi})|_{\text{(anti-)co-BRST}}}{\partial\bar{\theta}} = W_0 \end{aligned} \tag{26}$$

leads to the zero-forms of equations (21) and (22). Thus, the zero-forms in the expression for topological invariants find a geometrical interpretation as the translations for the local (but composite) superfields $(\Phi\bar{\Phi})(x, \theta, \bar{\theta})$ along the Grassmannian directions (θ and $\bar{\theta}$) of the supermanifold. By construction, these quantities are (anti-)BRST and (anti-)co-BRST invariant. From these expressions, one can always compute the rest of the topological invariants by exploiting the following recursion relations [5]:

$$\begin{aligned} s_b V_k &= \mathrm{d}V_{k-1} \qquad & \bar{s}_b \bar{V}_k &= \mathrm{d}\bar{V}_{k-1} \qquad & \mathrm{d} &= \mathrm{d}x^\mu \partial_\mu \\ s_b W_k &= \delta W_{k-1} \qquad & \bar{s}_b \bar{W}_k &= \delta \bar{W}_{k-1} \qquad & \delta &= \mathrm{i}\mathrm{d}x^\mu \varepsilon_{\mu\nu}\partial_\nu \end{aligned} \tag{27}$$

where $k = 1, 2$. The above relations are one of the key features for the existence of a TFT.

One of the central properties of a TFT is the lack of energy excitations in the physical sector of the theory. This happens because of the fact that when operator form of the Hamiltonian density ($T^{(00)}$) is sandwiched between two physical states, it yields zero (see, e.g., [3]). Thus, the form of the symmetric energy–momentum tensor $\left(T^{(s)}_{\mu\nu}\right)$ plays a very important role in this discussion. For the Lagrangian density ($\mathcal{L}_b$) of equation (1), the explicit form of the this symmetric tensor is [5–7]

$$\begin{aligned} T^{(s)}_{\mu\nu} = -\tfrac{1}{2}(\partial\cdot A)(\partial_\mu A_\nu + \partial_\nu A_\mu) - \tfrac{1}{2}E(\varepsilon_{\mu\rho}\partial_\nu A^\rho + \varepsilon_{\nu\rho}\partial_\mu A^\rho) \\ - \mathrm{i}\partial_\mu \bar{C}\partial_\nu C - \mathrm{i}\partial_\nu \bar{C}\partial_\mu C - \eta_{\mu\nu}\mathcal{L}_b. \end{aligned} \tag{28}$$

With the help of (17) (together with transformations (2) and (3) and equation (5)), it can be checked that the above equations can be written, modulo some total derivatives, as

$$T^{(s)}_{\mu\nu} = \left\{Q_b, S^{(1)}_{\mu\nu}\right\} + \left\{Q_d, S^{(2)}_{\mu\nu}\right\} \equiv \left\{\bar{Q}_b, \bar{S}^{(1)}_{\mu\nu}\right\} + \left\{\bar{Q}_d, \bar{S}^{(2)}_{\mu\nu}\right\} \tag{29}$$

where the local expressions for $S^{(1,2)}_{\mu\nu}$ and $\bar{S}^{(1,2)}_{\mu\nu}$ are

$$\begin{aligned} S^{(1)}_{\mu\nu} &= \tfrac{1}{2}[(\partial_\mu \bar{C})A_\nu + (\partial_\nu \bar{C})A_\mu + \eta_{\mu\nu}(\partial\cdot A)\bar{C}] \\ S^{(2)}_{\mu\nu} &= \tfrac{1}{2}[(\partial_\mu C)\varepsilon_{\nu\rho}A^\rho + (\partial_\nu C)\varepsilon_{\mu\rho}A^\rho - \eta_{\mu\nu}EC] \\ \bar{S}^{(1)}_{\mu\nu} &= -\tfrac{1}{2}[(\partial_\mu C)A_\nu + (\partial_\nu C)A_\mu + \eta_{\mu\nu}(\partial\cdot A)C] \\ \bar{S}^{(2)}_{\mu\nu} &= -\tfrac{1}{2}[(\partial_\mu \bar{C})\varepsilon_{\nu\rho}A^\rho + (\partial_\nu \bar{C})\varepsilon_{\mu\rho}A^\rho - \eta_{\mu\nu}E\bar{C}]. \end{aligned} \tag{30}$$

We can now exploit the finer details of the superfield expansions in (10a) and (15) to express the above S in terms of the superfields. Towards this goal, it is first essential to express T and P of (17) in the language of the superfields. It is straightforward to check, from the product of the odd superfields in (25), that

$$\begin{aligned} &\frac{\mathrm{i}}{2}\frac{\partial}{\partial\theta}[\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} = -\frac{1}{2}(\partial\cdot A)\bar{C} = T_2 \\ &-\frac{\mathrm{i}}{2}\frac{\partial}{\partial\bar{\theta}}[\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} = +\frac{1}{2}(\partial\cdot A)C = P_2 \\ &\frac{\mathrm{i}}{2}\frac{\partial}{\partial\theta}[\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}} = +\frac{1}{2}(EC) = T_1 \\ &-\frac{i}{2}\frac{\partial}{\partial\bar{\theta}}[\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}} = -\frac{1}{2}(E\bar{C}) = P_1 \end{aligned} \tag{31}$$

where the subscripts have the same interpretations as explained earlier (after equation (25)). It will be noticed that these T and P are the same (modulo some constant factors) as the zero-forms (26) in the topological invariants. Thus, these quantities have the same geometrical interpretation as the zero-forms of the topological invariants. Rest of the terms in $S^{(1,2)}_{\mu\nu}$ can be written, in terms of superfields, as

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial\theta}[B_\mu(x,\theta,\bar{\theta})B_\nu(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} = \frac{1}{2}(A_\mu\bar{R}_\nu + \bar{R}_\mu A_\nu)|_{\text{(anti-)BRST}} \\ &-\frac{1}{2}\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}\frac{\partial}{\partial\theta}[B^\rho(x,\theta,\bar{\theta})B^\sigma(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}} \\ &\qquad = -\frac{1}{2}\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}(A^\rho\bar{R}^\sigma + \bar{R}^\rho A^\sigma)|_{\text{(anti-)co-BRST.}} \end{aligned} \tag{32}$$

The rhs of the above equations can be expressed in terms of the gauge field A_μ and the (anti-)ghost fields $(\bar{C})C$ as

$$\tfrac{1}{2}[(\partial_\mu \bar{C})A_\nu + (\partial_\nu \bar{C})A_\mu] \qquad \text{and} \qquad \tfrac{1}{2}[(\partial_\mu C)\varepsilon_{\nu\rho}A^\rho + (\partial_\nu C)\varepsilon_{\mu\rho}A^\rho] \tag{33}$$

respectively. Here in equation (33), we have substituted the values of $\bar{R}$ from (8) and (13). This equation yields, modulo some total derivatives, the desired result. Ultimately, the expression for the $S^{(1,2)}_{\mu\nu}$ in terms of the superfields, are

$$\begin{aligned} S^{(1)}_{\mu\nu} &= \frac{1}{2}\frac{\partial}{\partial\theta}[B_\mu(x,\theta,\bar{\theta})B_\nu(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} - \frac{\mathrm{i}}{2}\eta_{\mu\nu}\frac{\partial}{\partial\theta}[\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} \\ S^{(2)}_{\mu\nu} &= -\frac{1}{2}\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}\frac{\partial}{\partial\theta}[B^\rho(x,\theta,\bar{\theta})B^\sigma(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}} \\ &\quad - \frac{\mathrm{i}}{2}\eta_{\mu\nu}\frac{\partial}{\partial\theta}[\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}}. \end{aligned} \tag{34}$$

Geometrically, the expression for $S^{(1)}_{\mu\nu}$ correspond to the translation of a second-rank tensor $B_\mu(x,\theta,\bar{\theta})B_\nu(x,\theta,\bar{\theta})$ (constructed by the even superfields) plus another second-rank tensor $\eta_{\mu\nu}\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta})$ (constructed by the odd superfields) along the θ-direction of the supermanifold. Similar interpretation can be attached to the local expression for $S^{(2)}_{\mu\nu}$. The local expressions for $\bar{S}^{(1,2)}_{\mu\nu}$ can also be computed in terms of the superfields. In fact, these depend on the derivative w.r.t. $\bar{\theta}$, as given below:

$$\begin{aligned} \bar{S}^{(1)}_{\mu\nu} &= -\frac{1}{2}\frac{\partial}{\partial\bar{\theta}}[B_\mu(x,\theta,\bar{\theta})B_\nu(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} + \frac{\mathrm{i}}{2}\eta_{\mu\nu}\frac{\partial}{\partial\bar{\theta}}[\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} \\ \bar{S}^{(2)}_{\mu\nu} &= +\frac{1}{2}\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}\frac{\partial}{\partial\bar{\theta}}[B^\rho(x,\theta,\bar{\theta})B^\sigma(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}} \\ &\quad + \frac{\mathrm{i}}{2}\eta_{\mu\nu}\frac{\partial}{\partial\bar{\theta}}[\Phi(x,\theta,\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}}. \end{aligned} \tag{35}$$

The geometrical interpretation in the language of the 'translations' can be given to the above expressions in the same way as that of their counterparts in (34). Finally, the expression for the symmetric energy–momentum tensor in (28) can be expressed in terms of the even superfields *alone* and the Grassmannian derivatives on them, as

$$\begin{aligned} T^{(s)}_{\mu\nu} &= \frac{\mathrm{i}}{2}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}[B_\mu(x,\theta,\bar{\theta})B_\nu(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} \\ &\quad - \frac{\mathrm{i}}{2}\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}[B^\rho(x,\theta,\bar{\theta})B^\sigma(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}} \\ &\quad - \frac{\mathrm{i}}{4}\eta_{\mu\nu}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}[B_\rho(x,\theta,\bar{\theta})B^\rho(x,\theta,\bar{\theta})]|_{\text{(anti-)BRST}} \\ &\quad - \frac{\mathrm{i}}{4}\eta_{\mu\nu}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}[B_\rho(x,\theta,\bar{\theta})B^\rho(x,\theta,\bar{\theta})]|_{\text{(anti-)co-BRST}} \end{aligned} \tag{36}$$

where the general expression for the first term in the above equation is

$$\frac{\mathrm{i}}{2}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}[B_\mu(x,\theta,\bar{\theta})B_\nu(x,\theta,\bar{\theta})] = -\frac{1}{2}(A_\mu S_\nu + S_\mu A_\nu) + \frac{\mathrm{i}}{2}(R_\mu\bar{R}_\nu - \bar{R}_\mu R_\nu). \tag{37}$$

In this derivation, the general form of the superfield expansion (6) has been used. To obtain the exact form of expression (28) for the symmetric energy–momentum tensor, one has to substitute in (37), the values of the extra secondary fields R_μ, $\bar{R}_\mu$, S_μ as quoted in equations (8) and (13), respectively. The other terms in (36) have been calculated earlier. In fact, in terms of the symmetry transformations, (36) can be recast as

$$T^{(s)}_{\mu\nu} = \frac{\mathrm{i}}{2}s_b\bar{s}_b\left(A_\mu A_\nu - \frac{1}{2}\eta_{\mu\nu}A_\rho A^\rho\right) - \frac{\mathrm{i}}{2}s_d\bar{s}_d\left(\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}A^\rho A^\sigma + \frac{1}{2}\eta_{\mu\nu}A_\rho A^\rho\right). \tag{38}$$

The geometrical interpretation for $T_{\mu\nu}^{(s)}$ in (36) can be provided in the same manner as the arguments and explanations given for the Lagrangian density after equation (19*b*). It appears to be an essential feature of a TFT that its symmetric energy–momentum tensor can be expressed as the $\theta\bar{\theta}$-component of a second-rank tensor that can be constructed by the even superfields of the theory. On this component, we apply the constraint conditions (8) and (13) that emerge after the imposition of the (dual) horizontality conditions.

It is gratifying to point out that, in the superfield formulation, the symmetric form of the energy–momentum tensor, the expressions for $T_{(1,2)}$, $P_{(1,2)}$ in (17), the expressions for $S_{\mu\nu}^{(1,2)}$ and $\bar{S}_{\mu\nu}^{(1,2)}$, the correct form of the topological invariants, etc, come out very naturally. Similarly, the form of the Lagrangian density turns out to be the Grassmannian derivatives on the Lorentz scalar $(B_\rho(x,\theta,\bar{\theta})B^\rho(x,\theta,\bar{\theta}))$ when we exploit the nilpotent (anti-)BRST and (anti-)co-BRST symmetries together with the generalized versions of the horizontality condition. To be more precise and more elaborate, it is the $\theta\bar{\theta}$-component of the above Lorentz scalar and the second-rank tensors: $B_\mu(x,\theta,\bar{\theta})B_\nu(x,\theta,\bar{\theta})$ and $\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}B^\rho(x,\theta,\bar{\theta})B^\sigma(x,\theta,\bar{\theta})$, that leads to the derivation of the Lagrangian density and the symmetric energy–momentum tensor. In this derivation, the generalized versions of horizontality condition w.r.t. the super cohomological operators $\tilde{\mathrm{d}}$ and $\tilde{\delta}$ play a very decisive role. Keeping in mind the geometrical interpretations for the (anti-)BRST charges $(\bar{Q}_b)Q_b$ and (anti-)co-BRST charges $(\bar{Q}_d)Q_d$ as the translation generators, it is obvious that the Lagrangian density in (17) (or its superfield analogue (19*a*)) and the energy–momentum tensor in (28) (or its superfield analogue in (36)) can be thought of as the translations of superfield versions (cf equations (18*a*) and (18*b*)) of the local composite fields $T_{(1,2)}(P_{(1,2)})$ and $S_{\mu\nu}^{(1,2)}\left(\bar{S}_{\mu\nu}^{(1,2)}\right)$ along the Grassmannian directions of the $(2+2)$-dimensional supermanifold. These properties are some of the key requirements for the existence of a TFT. Furthermore, it is also evident from (26) and (31) that the zero-forms of the topological invariants and P and T of (17) are nothing but the Grassmannian (θ and $\bar{\theta}$) components in the expansion of the superfields $\Phi\bar{\Phi}$. Geometrically, the zero-forms of the topological invariants are nothing but the translations of the local (but composite) fields $(\Phi\bar{\Phi})(x,\theta,\bar{\theta})$ along the θ and $\bar{\theta}$ directions of the $(2+2)$-dimensional supermanifold. It would be nice to apply this superfield formalism to the case of 2D self-interacting non-Abelian gauge theory and 4D free Abelian two-form gauge theory where the existence of nilpotent (anti-)BRST and (anti-)co-BRST symmetries have been demonstrated. Such studies might turn out to be useful in the context of topological string theories and topological gravity where, in contrast to the flat Minkowskian metric of our present discussion, a non-trivial (spacetime-dependent) metric is considered for the sake of generality. These are some of the issues that are under investigation and our results will be reported elsewhere [28].

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